

XVI. Newton's *Binomial Theorem legally demonstrated by Algebra*. By the Rev. William Sewell, A. M. Communicated by Sir Joseph Banks, Bart. K. B. P. R. S.

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LET m and n be any whole positive numbers; and $1 + \overline{x}^{\frac{m}{n}}$ a binomial to be expanded into a series, as $1 + Ax + Bx^2 + Cx^3 +$, &c. where A, B, C, D , &c. are the coefficients to be determined.

Assume $v^m = 1 + \overline{x}^{\frac{m}{n}} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 +$, &c.

And $z^m = 1 + \overline{y}^{\frac{m}{n}} = 1 + Ay + By^2 + Cy^3 + Dy^4 +$, &c.

Then will $v^n = 1 + x$, and $z^n = 1 + y \therefore v^n - z^n = x - y$.

And $v^m - z^m = A \times \overline{x - y} + B \times \overline{x^2 - y^2} + C \times \overline{x^3 - y^3} + D \times \overline{x^4 - y^4} +$, &c.

Consequently $\frac{v^m - z^m}{v^n - z^n} = A + B \times \overline{x + y} + C \times \overline{x^2 + xy + y^2} + D \times \overline{x^3 + x^2y + xy^2 + y^3} +$, &c. Now $v^m - z^m = \overline{v - z} \times : v^{m-1} + v^{m-2}z + v^{m-3}z^2 +$, &c.... z^{m-1} . Also $v^n - z^n = \overline{v - z} \times : v^{n-1} + v^{n-2}z + v^{n-3}z^2 +$, &c.... z^{n-1} . Therefore $\frac{v^m - z^m}{v^n - z^n}$ reduces to, and becomes $= \frac{v^{m-1} + v^{m-2}z + v^{m-3}z^2 +$, &c.... $z^{m-1}}{v^{n-1} + v^{n-2}z + v^{n-3}z^2 +$, &c.... $z^{n-1}}$
 $= A + B \times \overline{x + y} + C \times \overline{x^2 + xy + y^2} + D \times \overline{x^3 + x^2y + xy^2 + y^3} +$, &c.

The law is manifest; and it is likewise evident that the

numerator and denominator of the fraction, respectively terminate in m and n terms. Suppose then $x = y$; then will $v = z$; and our equation will become $\frac{mv^m - 1}{nv^n - 1}$, or $\frac{mv^m - 1}{n} = A + 2Bx + 3Cx^2 + 4Dx^3 + \&c.$

But $v^n = 1 + x$, therefore by multiplying we have $\frac{mv^m}{n} = A + \overline{A + 2Bx + 2B + 3Cx^2 + 3C + 4Dx^3 + \&c.}$ Or $v^m = \overline{1 + x}^{\frac{m}{n}} = \frac{nA}{m} + \frac{nA + 2nB}{m}x + \frac{2nB + 3nC}{m}x^2 + \frac{3nC + 4nD}{m}x^3 + \&c.$ Compare this with the assumed series, to which it is similar and equal, and it will be

$$\begin{aligned}\frac{nA}{m} &= 1 \\ \frac{nA + 2nB}{m} &= A \\ \frac{2nB + 3nC}{m} &= B, \\ \&c. &=, \&c.\end{aligned}$$

$$\therefore A = \frac{m}{n}; B = \frac{\overline{m - n}A}{1.2.n}; C = \frac{\overline{m - 2n}B}{1.2.3.n}; \&c.$$

Therefore $\overline{1 + x}^{\frac{m}{n}} = 1 + \frac{m}{n}x + \frac{m \times \overline{m - n}}{1.2.n^2}x^2 + \frac{m \times \overline{m - n} \times \overline{m - 2n}}{1.2.3.n^3}x^3 + \&c.$ the law is manifest, and agrees with the common form derived from other principles.

Sch. In the above investigation, it is obvious that unless m be a positive whole number, the numerator abovementioned does not terminate: it still remains, therefore, to shew how to derive the series when m is a negative whole number. In this case, the expression $(v^m - z^m)$ assumes this form, $\frac{1}{v^m} - \frac{1}{z^m}$, or its equal $\frac{z^m - v^m}{v^m z^m}$, which divided by $v^n - z^n$, as before, gives

$$\frac{1}{v^m z^m} \times \frac{z^m - v^m}{v^n - z^n} = \frac{1}{v^m z^m} \times \frac{\overline{-v - z} \times : v^{m-1} + v^{m-2}z + v^{m-3}z^2}{v - \times : v^{n-1} + v^{n-2}z + v^{n-3}z^2 +}, \&c. =$$

$$\frac{-1}{v^m x^{m-1}} \times \frac{v^{m-1} + v^{m-2}x + v^{m-3}x^2 + \&c.}{v^{m-1} + v^{m-2}x + v^{m-3}x^2 + \&c.} = (\text{when } v = x) - \frac{mv^{m-1}}{v^{2m} \times nv^{m-1}}$$

$$= -\frac{mv^{-m-n}}{n},$$
 which is the same as the expression $\left(\frac{mv^{m-n}}{n}\right)$ before derived with only the sign of m changed. The remainder of the process being the same as before, shews that the series is general, or extends to all cases, regard being had to the signs. Q. E. D.